TEACHERS' ORIENTATIONS TOWARD AND INTERACTIONS WITH STUDENTS' WRITING IN HIGH SCHOOL MATHEMATICS

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This report details a case study focused on different manifestations of teachers' orientations towards students' writing in high school mathematics. I make use of teacher interviews to unpack teachers' described orientations towards writing in mathematics, as well as their interpretations of interactions observed during recorded observations. These observations also illustrate the types of writing embedded in tasks enacted by these teachers and the nature of their interactions with students' writing during such tasks. Findings indicate participants' distinct understandings of "writing in math," show their flexibility in employing both teacher- and student-oriented interactions with student writing and suggest their desire to pursue more student-oriented interactions. Avenues for future research related to such findings are discussed.

Keywords: Communication, Instructional Activities and Practices, Teacher Beliefs, Teacher Noticing

Fostering students' effective communication of mathematical ideas is a key goal of educational reform movements and professional organizations in the United States (Common Core State Standards Initiative, 2010; National Council for Teachers of Mathematics, 2000). However, while supporting students' spoken communication has been a frequent focus of research in mathematics education research (Moschkovich, 2007; Munter et al., 2015; Webb et al., 2014), written communication has not received the same attention (Morgan, 1998; Pugalee, 2004). This relatively limited attention on writing is concerning because mathematical ideas are communicated not only through spoken language, but within and across the multiple sign systems of natural language (the written word), mathematical symbolism, and visual imagery – and these systems are often represented through written texts (O'Halloran, 2000, 2008). In other words, writing in mathematics is *multisemiotic*, or expressed within and across these multiple sign systems. Additionally, the language of mathematics itself evolved in part "as a response to the functions which were fulfilled symbolically and visually" (O'Halloran, 2008, p. 14). Therefore, mathematical meaning is not only being communicated multisemiotically through writing, but these sign systems also help inform the structure of spoken mathematical discourse. Student writing in the context of mathematics education thus warrants careful examination.

The overlapping nature of these multiple sign systems in mathematics also warrants the viewing of student writing expansively in the context of mathematics education. In this vein, I build from Draper and Siebert's (2010) framing of disciplinary literacy to broadly define writing in mathematics as the construction of any written text encompassing natural language, mathematical symbolism, or visual imagery that is used to understand or communicate mathematical ideas. This definition in turn conceives of a *text* as any representational object which is intended by its creator to communicate a meaning (Draper & Siebert, 2010; Schnotz et al., 2010).

My aim in the present study is to better understand teachers' orientations toward and interactions with students' writing in high school mathematics. In particular, I investigate instructional moments that arise at the intersection of students' written work and (spoken)

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discourse between students and teachers around such work, and address the ways that teachers conceive of students' writing in mathematics, attend to such writing, and interpret these interactions with such writing. I center this study on three questions: (1) Among a group of secondary mathematics teachers whose instruction is dialogically focused, what are their described orientations toward writing in school mathematics? (2) How do these teachers interact with their students' writing instructionally in relation to the writing expectations of the task? (3) How do these teachers describe their interpretation of students' writing and their interactions with students' writing?

Methods

This study employed an embedded multiple case study approach (Yin, 2018). Each of the teachers represented a case, while a subunit of teachers' observed interactions with students' writing was embedded within each of these three cases. Given that orientations arise at the intersection of beliefs, perceptions, and practices (Remillard & Bryans, 2004), this work made use of data from participant interviews, classroom observations, and collected samples of student written artifacts to form a nuanced reading of teachers' perceptions about student writing in mathematics and their observed interactions with such writing.

Participants and Setting

Three high school mathematics teachers in a Mid-Atlantic state participated in this study, with pseudonyms Mrs. Taylor, Mrs. Hudson, and Mrs. Barnett. All three teachers' instruction had previously been coded as dialogically focused during their participation in the SMiLES (Secondary Mathematics in-the-moment Longitudinal Engagement Study) project, which had used classroom observation data to identify potentially productive mathematics teaching practices (see Jansen et al., 2021). "Dialogically focused" instruction in this case entailed the use of (1) high-level, open-ended tasks, (2) opportunities for sharing multiple representations or strategies (e.g., graphs, tables, etc.), and (3) student discourse (Henningsen & Stein, 1997; Munter et al., 2015). Given the present study's focus on the relationships between student writing and spoken discourse, dialogically focused instruction was conjectured to be an opportune setting under which to potentially capture teachers interacting with students' writing.

Measures and Analysis

The data for this study came from two teacher interviews (pre- and post-observation) for each participant, recorded classroom observations (four for each participant during the Spring 2022 school year), and collected samples of student writing (captured during classroom observations). The pre-observation teacher interview drew on items adapted from Cantrell and colleagues (2008) meant to capture participants' described orientations towards writing in mathematics (the subject of the first research question). In particular, these questions addressed the extent to which participants descriptions about writing in mathematics are multisemiotic (i.e., consider different written sign systems) and whether participants oriented towards a writing-to-learn perspective (i.e., that writing is not merely a way to summarize what has already been learned, but that learning can occur through the practice of writing in mathematics; Connolly, 1989).

The second interview built off of this pre-observation interview, first allowing participants to member-check (Candela, 2019; Creswell & Clark, 2017) profiles created to summarize their described orientations about writing from the previous interview. Teachers were also asked to provide their noticings about samples of student writing captured during observations (see Goldsmith & Seago, 2011; Jacobs et al., 2010) and to engage in video viewing sessions (Erickson, 2007) of multiple interactions that they had with student writing as captured from

their recorded classroom observations (described further below). Selection of these instructional video clips followed criteria similar to Sherin et al. (2009), namely that the chosen clips showed evidence of student thinking that was easily understandable and focuses on substantive mathematical ideas. In support of this, I particularly drew on clips that aligned with the captured samples of student writing and which addressed different types of interactions with student writing (described below). Participant responses to these interview components were used to address the third research question.

For the second research question, four recorded observations for each teacher were coded along dimensions of (1) the type of writing embedded in the instructional tasks enacted during the lesson and (2) the type of teacher interaction with students' writing. The categories of writing task was based on the work of Casa and colleagues (2016), and categorized the writing as either exploratory (to make personal sense of a situation or problem), procedural/symbolic (to calculate a numeric or algebraic solution by using procedures with mathematical symbolism), informative (to describe something), exclusive explanatory (to explain something, students executing "already-known" ideas), or inclusive explanatory (to explain something, students as active participants in forming ideas). The differentiation among types of explanatory writing tasks as either "exclusive" or "inclusive" was derived from the curriculum-focused work of Rotman (1988) and Herbel-Eisenmann (2007), where differences were noted in whether mathematical task prompts framed the reader (e.g., the student completing the task) as merely executing a defined set of statements to arrive at an already-known conclusion (exclusive, e.g., "Prove that $x = 7$ is a valid solution to the given equation.") or as an active participant in the living activity of "doing" mathematics (inclusive, "How might we find a solution to the following equation?"). The distinction between these explanatory types of writing were pursued under the conjecture that the framing of the prompt could foster different types of teacher interactions with the writing that students might produce during their engagement with such tasks.

The types of teacher interactions with students' writing, shown in Table 1, were based on literature concerning teachers' responses to student thinking in mathematics, as well as past research on instructional approaches to questioning. Firstly, an interaction could entail a teacher-driven response to a students' writing. This could involve a purely *evaluative* response (e.g., "Good job, that is correct" or "No, the solution is 12"), or a response that evaluates the student writing but also *describes* how the teacher sees (or do not see) their writing as communicating mathematical meaning (e.g., "I appreciate that you wrote out each step as you applied different operations to solve this equation, as that helps me understand how you arrived at your solution."). This latter type of interaction could be seen as a type of judicious telling (Smith, 1996) that uses students' own written or spoken ideas in order to focus those students onto specific mathematical ideas relevant to the learning goals (see Smith et al., 2023).

However, teachers could also more actively involve students in these interactions through a process that van Es and Sherin (2021) call *shaping*, which "involves teachers constructing interactions, in the midst of noticing, to gain access to additional information that further supports their noticing" (p. 23). This is aligned with what Boston et al. (2017) describe as *assessing* and *advancing* questions, with the former entailing the teacher assessing students' understanding of the mathematics behind their written work (e.g., "Where did this '*4' come from in your equation here?") and the latter entailing a noticing or wonderment meant to extend students' thinking (and writing) towards the learning goal (e.g., "Why did you draw the slope of this function as steeper than this other function? What about the function equation informs you how steep the slope might be when you graph it?"). With advancing questions – which I expand

in this study to include suggestive statements that serve a similar point as questions (e.g., "See what happens to the function rule when you change the slope.") – the teacher is meant to then leave the student(s) to continue their thinking independently.

Table 1: Types of Teacher Interactions with Students' Writing in Mathematics

Type of interaction	Description
Evaluative Response	The teacher addresses a student's written text *without* then explaining how they see the writing as communicating mathematical meaning.
Descriptive Response	The teacher addresses a student's written text and/or states how they see the writing as communicating mathematical meaning.
Assessing Question(s) + Response	The teacher inquires the student about the mathematical meaning of their written text and then states how they see the writing as communicating mathematical meaning.
Response + Advancing Question(s)	The teacher states how they see the students' written text as communicating mathematical meaning and then ends the interaction with a question or suggested next step to advance or revise the students' writing.
Assessing Question(s) + Response + Advancing Question(s)	The teacher inquires the student about the mathematical meaning of their written text, states how they see the writing as communicating mathematical meaning, and then ends the interaction with a question or suggested next step to advance or revise the students' writing.

Such question-oriented interactions represent a more dialogic and student-focused approach to interacting with students' writing, as they allow students to retain at least some of their own voice in relation to the written work that they have produced, and to more accurately expand upon the mathematical meaning behind the written text that they had produced. I also anticipated and encountered mixed combinations of these sorts of questions with more teacher-focused evaluative and descriptive responses. For instance, a teacher may give descriptive feedback and follow this with an advancing question, or might ask an assessing question before steering the interaction towards an evaluative response. As such, Table 1 represents combinations of how assessing and advancing questions arose in the data.

Results

Teacher interviews indicated that the participants had distinct and nuanced described orientations towards writing in mathematics, with a clear emphasis on natural language writing (e.g., written explanations or justifications) but openness to multisemiotic perspectives towards writing and writing-to-learn. Classroom observation data indicated a variety of interactions with writing across a range of writing tasks, albeit with the most common interactions being teacher-drive (i.e., evaluative or descriptive responses) with procedural/symbolic tasks. Teachers' descriptions of selected interactions, however, did indicate that the participants had a desire for more student-centered interactions and exploratory or explanatory writing tasks.

Described Orientations Towards Writing in Mathematics

The participants' described orientations towards writing in mathematics are summarized in Table 2. This table shows the type of sign system(s) emphasized by participants in their interviews (i.e., "Semiotic Orientation") and the ways in which teachers affirmed a writing-to-learn (Connolly, 1989; Morgan, 1998) perspective, with writing framed as an instructional tool for supporting the learning of conceptual ideas (i.e., "Learning Orientation). All three of the participants described – or were receptive to – writing as arising through multiple sign systems in the doing and learning of mathematics (a multisemiotic perspective), and all participants were receptive to a writing-to-learn perspective. However, participants at times noted exceptions or reservations towards such receptive attitudes.

Table 2: Teachers' Described Orientations Towards Writing in Mathematics

Name	Semiotic Orientation	Learning Orientation
Mrs. Taylor	Natural language (written explanations) Receptive to a multisemiotic perspective*	Receptive to writing-to-learn*
Mrs. Hudson	Natural language (academic vocabulary) Receptive to a multisemiotic perspective*	Receptive to writing-to-learn
Mrs. Barnett	Writing as multisemiotic (words, visuals, symbols)	Receptive to writing-to-learn*

* (with exceptions or reservations)

Mrs. Taylor largely centered her descriptions about writing in mathematics in terms of writing to justify or explain mathematical processes and solutions through natural language (i.e., the written word). She saw such writing as complementary to (but distinct from) more symbolically focused, procedural tasks. While her descriptions did not emphasize connections across different sign systems of writing, Mrs. Taylor did state that more symbolic, procedural writing done by students and more explanatory, natural language writing were "both valuable" in mathematics. She also noted that, although students can learn mathematical ideas through the construction of written explanations, this is "the part that goes when you're on a time crunch." Thus, while she was sympathetic to a writing-to-learn perspective, she did not describe writing (via written explanations or justifications) as crucial to the learning process.

Mrs. Hudson's descriptions of writing centered largely on the idea of academic vocabulary (e.g., "monomial"). She noted how, because the language that we use to describe mathematical concepts or representations is rooted in the nature of such ideas, that by "breaking down English words and translating" them (e.g., discussing the "mono" in "monomial"), students can recognize "how they [the words] affect the math world." Thus, while her descriptions focused on natural language writing, Mrs. Hudson did see such writing as helping to develop students' understanding of other sign systems. She specifically described how, by frequently and publicly making use of students' written descriptions or explanations of relevant mathematical ideas, she saw writing as a way for students to gain comfort in both writing and speaking about mathematics with their peers, focus away from "if I'm correct or not," and value what they can learn from their peers' language. As such, she felt that writing "should go throughout" the learning cycle, including "warm up" activities, "rough draft" math, checks for understanding, and "exit ticket" routines, indicating a receptive stance towards writing-to-learn.

Mrs. Barnett held the most explicitly multisemiotic description of writing in mathematics, stating that "math is a form of communication in and of itself," where "I have this idea that I want to communicate to people, this pattern that I've noticed…and I just need to be able to demonstrate it for someone else in however way I see fit with words, visuals, [or] symbols." As mathematics and writing were both seen as forms of communication, Mrs. Barnett held that they were closely "intertwined" with one another. Because of this, she noted that "you could justify it [writing] anywhere in the learning process," although she also stated that her immediate thought was that writing best occurs at the end of a learning sequence "to summarize" what was learned. Such responses indicated that Mrs. Barnett was receptive towards writing-to-learn but held some reservations with that perspective.

Observed Teacher Interactions with Students' Writing in Mathematics

For each participant, four 90-minute lessons were recorded across the Spring 2022 semester and analyzed to identify types of writing tasks used by the teachers and types of interactions that they had with students' writing. These results are shown below in Table 3.

Table 3: Types of Writing Tasks and Teacher Interactions with Student Writing

	Procedural/ Symbolic	Exploratory	Informative	Explanatory (exclusive)	Explanatory (inclusive)	**TOTAL**
Evaluative Response	39 (6/23/10)	0 (0/0/0)	12 (3/4/5)	6 (5/0/1)	19 (7/12/0)	**76 (21/39/16)**
Descriptive Response	42 (3/12/27)	2 (0/0/2)	3 (1/2/0)	10 (8/1/1)	13 (8/5/0)	**71 (20/21/30)**
Assessing Question(s) + Response	11 (2/5/4)	3 (0/0/3)	1 (1/0/0)	2 (2/0/0)	6 (4/2/0)	**22 (9/6/7)**
Response + Advancing Question(s)	3 (0/2/1)	3 (1/0/2)	0 (0/0/0)	3 (2/1/0)	4 (3/1/0)	**13 (6/4/3)**
Assessing + Response + Advancing Questions	2 (0/1/1)	0 (0/0/0)	1 (1/0/0)	2 (2/0/0)	2 (2/0/0)	**7 (5/1/1)**
TOTAL	**97 (11/43/43)**	**8 (1/0/7)**	**17 (6/6/5)**	**23 (19/2/2)**	**44 (24/20/0)**	**189 (61/71/57)**

Note. Results show totals followed by individual teacher counts (Taylor/Hudson/Barnett) below.

These results show a variety of interactions that occurred across a range of writing tasks. For Mrs. Hudson and Mrs. Barnett, most of these interactions occurred during procedural/symbolic tasks (43 interactions each), while for Mrs. Taylor, a plurality of interactions occurred during explanatory (inclusive) tasks (24 interactions), followed by explanatory (exclusive) tasks (19 interactions). The results also show that evaluative and descriptive interactions (i.e., teacher-oriented interactions) composed the majority of interactions observed in every participant's lessons (41 out of 61 interactions for Mrs. Taylor, 60 out of 71 interactions for Mrs. Hudson, and 46 out of 57 interactions for Mrs. Barnett). Given the briefness of many such interactions (e.g.,

"That solution is correct" could be coded as an evaluative interaction), it is unsurprising that these interactions would be more frequent than those that required more extensive open-ended discourse (i.e., assessing and/or advancing questions) between the teacher and student.

Interestingly, the results also show that each participant did engage in student-oriented interactions (i.e., those that used assessing and/or advancing questions), and that such interactions occurred across the range of writing expectations embedded within tasks. Every teacher, for instance, was observed using such questions multiple times with procedural/ symbolic tasks. The results also show how, even as students were working on tasks that required explanatory writing, teachers continued to give quick evaluative or descriptive feedback to students in addition engaging them at times with assessing and/or advancing questions.

Teachers' Described Interpretations of Students' Writing in Mathematics

In their post-observation interviews, teachers were asked to interpret samples of students' writing collected during the observations and captured examples of their interactions with students' writing from those same lessons. These interpretations inform the third research question, and demonstrate important connections between participants' described orientations and observed interactions with student writing.

Mrs. Taylor's reactions to samples of student writing are in line with her focus on writing as a form of explanation and justification. She was asked to interpret samples of posters that student groups created to "prove" the classification of a quadrilateral (e.g., parallelogram) based on four given coordinates that represented the location of the quadrilateral's vertices. She mentioned how, although the different student groups "got full credit because they did a fabulous job showing me their work and everything like that," she wished that she had seen more students "showing me how you came up with those answers." This again showed the way that she differentiated between more symbolic writing and written explanations.

Mrs. Hudson's interpretations of students' writing emphasized her previously described focus on building student comfort with sharing their mathematical thinking through both writing and speaking. For instance, she was asked to reflect on students' writing from an activity where they were solving for unknown exponents (e.g., $3^b * 3^7 = 3^{11}$) and prompted to write out their solutions and reasoning through an app that Mrs. Hudson then used to display and comment on student responses whole class. She described how, even for students' explanations that were imprecise or used incorrect grammar, she was "ok with that" because students were still "able to verbalize their thoughts or write out their thoughts and reasoning in a way that makes sense to them, and then we transition into some vocab that applies to what they're saying." This quote also alludes to Mrs. Hudson's previously stated emphasis on academic vocabulary.

Mrs. Barnett's interpretation of student writing samples appeared to reflect her described orientation towards writing as a form of communication. She reflected on a warm-up activity she had enacted where students were asked to describe what they noticed about a graph of the logarithmic function $f(x) = \log_b(x)$, the exponential function $g(x) = b^x$, and the line $y = x$. This was done to demonstrate the inverse relationship between logarithmic and exponential functions, and these responses were then displayed and discussed whole group. She recounted how she used to "get very just fixated on converting from logarithmic to exponential form," and saw this activity as a way for students to grapple with seeing patterns across different sign systems. This emphasis on translating within and across sign systems appeared in line with Mrs. Barnett's previous multisemiotic descriptions.

Mrs. Barnett was then asked to reflect on a recording of an interaction she had with students around this task. Her response was revealing in how she thought about her work with students

around their writing and aligned with similar comments made by the other participants. The interaction in question was a descriptive response, but Mrs. Barnett stated how "I wish I had asked the students to make sense of each other's responses instead of reading through them myself." This indicated that she was aware of the teacher-oriented nature of the interaction, and expressed a desire for more student ownership and voice during such interactions.

Discussion and Conclusions

This study indicates the multiple dimensions through which teachers orient themselves towards writing in mathematics. Because teacher orientations arise at an intersection of beliefs, perceptions, and practices (Remillard & Bryans, 2004), this investigation is particularly helpful in shedding light on how teachers' descriptions of their orientation towards writing in mathematics compare to their observed interactions with students' writing and their interpretations of such interactions. Because this study employed a case study approach (Yin, 2018), the results are not generalizable. However, the categories devised for analyzing the observation data and the overarching study results offer a framework for future investigations. Importantly, cross-case analysis indicates some salient themes regarding how these teachers orient themselves towards writing in mathematics.

In terms of described orientations, each participants had a distinct semiotic orientation towards writing, from a focus on written explanations and justifications (Mrs. Taylor), to a focus on academic vocabulary (Mrs. Hudson), to a focus on communicating across sign systems (Mrs. Barnett). Although only Mrs. Barnett initially described writing in mathematics across multiple sign systems, all participants were receptive to a multisemiotic (O'Halloran, 2000, 2008) perspective on writing. Each participant was also receptive to a writing-to-learn (Connolly, 1989; Morgan, 1998) perspective, although Mrs. Taylor indicated that this was not essential for teaching mathematics and Mrs. Barnett noted that she generally thinks of writing as a way to summarize what has already been learned.

Although the teachers were receptive to these multisemiotic, writing-to-learn perspectives of writing in mathematics, their observed instruction focused primarily on procedural/symbolic tasks, and most of their interactions were evaluative or descriptive in nature. However, across all types of writing tasks teachers were still observed using assessing and/or advancing questions. This suggests that such teachers play a critical role as a facilitator in determining how students' writing is interpreted and interacted with during instruction. In other words, these teachers are able to enact even closed-ended, procedural tasks in ways that, at times, allow for student-oriented discourse around their written work to arise.

Participants' recognition of teacher-oriented interactions as being "missed opportunities" for student discourse also showed how these teachers could identify different types of interactions with students writing and signaled their preference for more student-oriented interactions. Given that participants' instruction had previously been found to be dialogically oriented (Munter et al., 2015), this could indicate the value that such teachers place upon student discourse. As such, supporting such teachers in building bridges between their orientations towards discourse and orientations towards writing in mathematics may be a compelling way to foster more consistent student-oriented interactions with students' written work.

Ultimately, participants in this study held distinct orientations towards "writing in math." However, their instruction and reflections suggested a strong desire to foster student ownership in the doing of and discourse around mathematics through such writing. By highlighting the connections between written and spoken communication in mathematics, as well as the multiple

sign systems through which we communicate such ideas, such teachers might be better able to foster students' effective communication in the learning and doing of mathematics.

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